THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH3070 Introduction to Topology 2017-2018
Solution of Tutorial Classwork 6

1. ($\Rightarrow$) Let $\{U_\alpha\}_\alpha$ be an open cover of the Z in $(X, \mathfrak{T})$. Then $\{U_\alpha \cap Y\}_\alpha$ is an open cover of Z in $(Y, \mathfrak{T}_Y)$. By compactness of Z in $(Y, \mathfrak{T}_Y)$, there exists a finite subcover $\{U_i \cap Y\}_{i=1}^n$ of Z. Since $Z \subset Y$, $\{U_i\}_{i=1}^n$ is a finite subcover of Z in $(X, \mathfrak{T})$. Hence Z is compact in $(X, \mathfrak{T})$.

   ($\Leftarrow$) Let $\{V_\alpha\}_\alpha$ be an open cover of the Z in $(Y, \mathfrak{T}|_Y)$. Since $V_\alpha \in \mathfrak{T}|_Y$, we have $V_\alpha = W_\alpha \cap Y$ for some $W_\alpha \in \mathfrak{T}$. Then $\{W_\alpha\}_\alpha$ is an open cover of Z in $(X, \mathfrak{T})$. By compactness of Z in $(X, \mathfrak{T})$, there exists a finite subcover $\{W_i\}_{i=1}^n$ of Z. Since $Z \subset Y$, $\{W_i \cap Y\}_{i=1}^n$ is a finite subcover of Z in $(Y, \mathfrak{T}|_Y)$. Hence Z is compact in $(Y, \mathfrak{T}|_Y)$.

2. By Q)1), it suffices to show that $C_1 \cap C_2$ is compact in $(C_1, \mathfrak{T}|_{C_1})$. Note that C_1 is compact. Hence it suffices to show that $C_1 \cap C_2$ is closed in $(C_1, \mathfrak{T}|_{C_1})$.

   Since C_1 and C_2 are compact sets in the Hausdorff space X, C_1 and C_2 are closed in X. Therefore, $C_1 \cap C_2$ is closed in X. This implies that $C_1 \cap C_2$ is closed in C_1. Hence $C_1 \cap C_2$ is compact.

3. * For each $n \in \mathbb{N}$, consider the open cover $\{B(x, \frac{1}{n}) \mid x \in X\}$. By compactness of X, there exists a finite subcover $\{B(x_l^{(n)}, \frac{1}{n})\}_{l=1}^{k_n}$ of X. Consider the set $D = \{x_l^{(n)} \mid n \in \mathbb{N}, l = 1, 2, \ldots, k_n\}$. We want to show that D is a countable dense set. Since D is countable union of finite sets, it is countable. Furthermore, pick any point $x \in X$ and consider any open ball $B(x, r)$ centered at x with radius r. Choose n_0 such that $\frac{1}{n_0} < r$. Consider the open cover $\{B(x_l^{(n_0)}, \frac{1}{n_0})\}_{l=1}^{k_{n_0}}$ of X. Since it is an open cover, there exists some point $x_{l_0}^{(n)}$ such that $x \in B(x_{l_0}^{(n)}, \frac{1}{n_0})$. Hence $d(x, x_{l_0}^{(n)}) < \frac{1}{n_0} < r$ and $x_{l_0}^{(n)} \in B(x, r)$. This shows that D is a dense set.